Computers and Mathematics with Applications 56 (2008) 1861–1875

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Generalized Eckstein–Bertsekas proximal point algorithm based on *A*-maximal monotonicity design

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ARTICLE INFO

Article history:
Received 7 March 2008
Accepted 17 April 2008

Keywords:
Variational inclusions
General maximal monotone mapping
A-maximal monotone mapping
Generalized Eckstein–Bertsekas proximal point algorithm
Generalized resolvent operator

ABSTRACT

A general design for the Eckstein–Bertsekas proximal point algorithm, using the notion of the *A*-maximal monotonicity, is developed. Convergence analysis for the generalized Eckstein–Bertsekas proximal point algorithm in the context of solving a class of nonlinear inclusion problems is explored. Some auxiliary results of interest involving *A*-maximal monotone mappings are also included. The obtained results generalize investigations on general maximal monotonicity and beyond.

1. Introduction

The Eckstein–Bertsekas proximal point algorithm [1] generalizes the proximal point algorithm introduced by Rockafellar [2,3], while investigating nonlinear programming. This in turn generalizes the algorithm of Martinet [4] for convex programming. It seems that a general class of problems of variational character, including minimization or maximization of functions, variational inequality problems, and minimax problems can be unified into this form. Furthermore, it was shown that the Douglas–Rachford splitting method [5] for convex programming was in fact a special case of the Eckstein–Bertsekas proximal point algorithm, and it prompted a greater degree of unification and generalization for a general class of algorithms for convex programming.

Let X be a real Hilbert space with the norm $\|.\|$ and the inner product $\langle ., .\rangle$. We consider the inclusion problem: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

In [2], Rockafellar examined the general convergence analysis and rate of convergence for an algorithm (referred to as the proximal point algorithm in literature) in the context of solving (1) by showing, when M is maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by the iterative procedure

$$x^{k+1} \approx P_k(x^k), \tag{2}$$

converges weakly to a solution to (1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $P_k = (I + c_k M)^{-1}$ for a sequence $\{c_k\}$ of positive real numbers that are bounded away from zero. Then from (2), we conclude that x^{k+1} is an approximate solution to the inclusion problem

$$0 \in M(x) + c_k^{-1}(x - x^k). \tag{3}$$

The proof clearly follows from the definition of $P_k = (I + c_k M)^{-1}$.

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doi:10.1016/j.camwa.2008.04.026

Based on [1], we state the Eckstein–Bertsekas proximal point algorithm:

Algorithm 1.1. Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k w^k \quad \forall k \geq 0, \tag{4}$$

where w^k is such that

$$\|w^k - (I + c_k M)^{-1}(x^k)\| \leq \epsilon_k \quad \forall k \geq 0,$$

and

$$\{\epsilon_k\},\ \{\alpha_k\} \text{ and } \{c_k\} \subseteq [0, \infty)$$

are scalar sequences.

Eckstein and Bertsekas [1] applied Algorithm 1.1 to approximating a weak solution to variational inclusion problem (1).

Theorem 1.1 (*[1, Theorem 3]*). *Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by Algorithm 1.1. If the scalar sequences $\{\epsilon_k\}$, $\{\alpha_k\}$ and $\{c_k\}$ satisfy*

$$E_1 = \sum_{k=0}^{\infty} \epsilon_k < \infty, \qquad \triangle_1 = \inf \alpha_k > 0, \qquad \triangle_2 = \sup \alpha_k < 2, \quad \textit{and} \quad c = \inf c_k > 0,$$

then the sequence $\{x^k\}$ converges weakly to a zero of M.

The convergence analysis for Algorithm 1.1 is based on the notion of the firm nonexpansiveness of the resolvent operator $(I + c_k M)^{-1}$. General maximal monotonicity has been a powerful framework for studying convex programming as well as variational inequalities. Based on several investigations, it turned out that one of the fundamental algorithms used for solving these problems was the proximal point algorithm. Eckstein and Bertsekas [1] has demonstrated that much of the theory of Algorithm 1.1 and other variant algorithms can be carried over to the Douglas–Rachford splitting method and its specializations, like the alternating direction method of multipliers.

Recently Pennanen [6] studied a localized version of the maximal monotonicity, and has over-relaxed the Eckstein–Bertsekas proximal point algorithm to show that the sequence converges strongly to a unique solution of (1). There happens to be a further discussion of the local convergence of multiplier methods for a general class of problems in [6]. This presents the specializations as new convergence results for multiplier methods for nonmonotone variational inequalities and nonconvex nonlinear programming.

While studying nonlinear convex programming, Rockafellar [3] introduced the following proximal method of multipliers.

Algorithm 1.2. Step 1. Choose (x_0, y_0) and set $k = 0$.

Step 2. Minimize

$$\phi_k(x) := f_0(x) + \frac{1}{2c_k}\|x - x_k\|^2 + \frac{1}{2c_k} d_k(y_k + c_k F(x))^2 \quad \text{for } x_{k+1},$$

where f_i are real-valued C^2 functions, C a closed convex subset of a Hilbert space X, and

$$F(x) = (f_1(x), \ldots, f_m(x)).$$

Step 3. Set

$$y^i_{k+1} = y^i_k + c_k f_i(x_{k+1}) \quad \text{for } i = 1, \ldots, r,$$
$$y^i_{k+1} = \max\{y^i_k + c_k f_i(x_{k+1}), 0\} \quad \text{for } i = r+1, \ldots, m,$$

$k = k + 1$ and go to Step 2.

Clearly, proximal point method of multipliers differs from the classical method of multipliers in just one sense that it has an additional regularizing term $\frac{1}{2c_k}\|x - x_k\|^2$ in ϕ_k, while in the case of convex programming [3], this term allows us to derive better convergence properties than for the ordinary method of multipliers.

Recently, the author [7–10] generalized the notion of maximal monotone mappings, including the notion of the H-maximal monotonicity introduced and studied by Fang and Huang [11] in the context of approximating the solution of an inclusion problem using the resolvent operator technique. The generalized resolvent operator techniques have been applied to other problems from other fields such as equilibria problems in economics, optimization and control theory, operations research, and mathematical programming. There is a vast literature on the applications of the generalized resolvent operators and techniques to solving the variational inclusion problems involving several classes of monotone mappings, including strongly monotone and relaxed cocoercive mappings. For more details on the resolvent operator techniques and related literature, we refer the reader to [1–45].

In this paper, we intend to develop a general framework for the generalized Eckstein–Bertsekas proximal point algorithm based on the notion of A-maximal monotonicity design [7–9], and then apply it to approximating a solution to a general class of nonlinear inclusion problems involving A-maximal monotone mappings in a Hilbert space setting. Then we examine the convergence analysis of the generalized Eckstein–Bertsekas proximal point algorithm for solving a class of nonlinear inclusion problems. Furthermore, some auxiliary results on the generalized firm nonexpansiveness and generalized resolvent mappings are derived. The results, thus obtained, are general in nature and application-oriented as well.

The organization of contents is as follows. Section 1 deals with a historical development of the proximal point algorithm and Eckstein–Bertsekas proximal point algorithm (based on the celebrated work of Rockafellar [2,3]) in conjunction with general maximal monotonicity, and with the approximation solvability of a class of nonlinear inclusion problems using the convergence analysis for the Eckstein–Bertsekas proximal point algorithm. Section 2 is concerned with some auxiliary results related to the main results on hand. In Section 3, a general framework for Eckstein–Bertsekas proximal point algorithm is introduced, and then it is applied to approximating the solution to inclusion problem (1).

2. *A*-Maximal monotonicity and applications

This section deals with some results based on basic properties of A-maximal monotonicity, and the introduction and derivation of results on A-monotonicity and generalized firm nonexpansiveness. Let X denote a real Hilbert space with the norm $\|.\|$ and inner product $\langle .,.\rangle$. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$\mathrm{dom}(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

$\mathrm{dom}(M) = X$, shall denote the full domain of M, and the range of M is defined by

$$\mathrm{range}(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{(x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) (r)-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M).$$

(ii) (1)-strongly monotone if

$$\langle u^* - v^*, u - v\rangle \geq \|u - v\|^2 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M).$$

(iii) (r)-strongly pseudomonotone if

$$\langle v^*, u - v\rangle \geq 0$$

implies

$$\langle u^*, u - v\rangle \geq r\|u - v\|^2 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M).$$

(iv) pseudomonotone if

$$\langle v^*, u - v\rangle \geq 0$$

implies

$$\langle u^*, u - v\rangle \geq 0 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M).$$

(v) (m)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M)$$

(vi) (c)-cocoercive if there is a positive constant c such that

$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2 \quad \forall\, (u, u^*), (v, v^*) \in \mathrm{graph}(M).$$

Definition 2.2. A mapping $M : X \to 2^X$ is said to be maximal (m)-relaxed monotone if

(i) M is (m)-relaxed monotone,
(ii) For $(u, u^*) \in X \times X$, and
$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \quad \forall (v, v^*) \in \operatorname{graph}(M),$$
we have $u^* \in M(u)$.

Definition 2.3. Let $M : X \to 2^X$ be a mapping on X. The map M is said to be:

(i) Nonexpansive if
$$\|u^* - v^*\| \leq \|u - v\| \quad \forall (u, u^*), (v, v^*) \in \operatorname{graph}(M).$$
(ii) Firmly nonexpansive if
$$\|u^* - v^*\|^2 \leq \langle u^* - v^*, u - v\rangle \quad \forall (u, u^*), (v, v^*) \in \operatorname{graph}(M).$$
(iii) (c)-Firmly nonexpansive if there exists a constant $c > 0$ such that
$$\|u^* - v^*\|^2 \leq \|u - v\|^2 - c\|u - v - (u^* - v^*)\|^2 \quad \forall (u, u^*), (v, v^*) \in \operatorname{graph}(M).$$
(iv) (c)-Firmly nonexpansive if there exists a constant $c > 0$ such that
$$\|u^* - v^*\|^2 \leq c\langle u^* - v^*, u - v\rangle \quad \forall (u, u^*), (v, v^*) \in \operatorname{graph}(M).$$

Definition 2.4 ([7]). Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be A-maximal monotone if:

(i) M is (m)-relaxed monotone.
(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.5 ([11]). Let $H : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be H-maximal monotone if:

(i) M is monotone.
(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Proposition 2.1 ([7]). *Let $A : X \to X$ be a (r)-strongly monotone single-valued mapping, and let $M : X \to 2^X$ be an A-maximal monotone mapping. Then $A + \rho M$ is maximal monotone for $\rho > 0$.*

Proposition 2.2 ([11]). *Let $H : X \to X$ be a (r)-strongly monotone single-valued mapping and let $M : X \to 2^X$ be an H-maximal monotone mapping. Then $H + \rho M$ is maximal monotone for $\rho > 0$.*

Proposition 2.3. *Let $H : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an H- maximal monotone mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued.*

Definition 2.6. Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-maximal monotone mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by
$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Definition 2.7. Let $H : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an H-maximal monotone mapping. Then the generalized resolvent operator $J^M_{\rho,H} : X \to X$ is defined by
$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u).$$

Definition 2.8. Let $A, T : X \to X$ be two mappings on X. Then map T is said to be:

(i) Monotone with respect to A if
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq 0 \quad \forall (x, y) \in X.$$
(ii) (r)-*strongly* monotone with respect to A if there exists a positive constant r such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq r\|x - y\|^2 \quad \forall (x, y) \in X.$$

(iii) (γ, α)-relaxed cocoercive with respect to A if there exist positive constants γ and α such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq -\gamma \|T(x) - T(y)\|^2 + \alpha\|x - y\|^2 \quad \forall (x, y) \in X.$$

Definition 2.9. Let $H, T : X \to X$ be two mappings. Then map T is said to be:

(i) Monotone with respect to H if

$$\langle T(x) - T(y), H(x) - H(y)\rangle \geq 0 \quad \forall\, (x, y) \in X.$$

(ii) (r)-strongly monotone with respect to H if there exists a positive constant r such that

$$\langle T(x) - T(y), H(x) - H(y)\rangle \geq r\|x - y\|^2 \quad \forall\, (x, y) \in X.$$

(iii) (γ, α)-relaxed cocoercive with respect to H if there exist positive constants γ and α such that

$$\langle T(x) - T(y), H(x) - H(y)\rangle \geq -\gamma \|T(x) - T(y)\|^2 + \alpha\|x - y\|^2 \quad \forall (x, y) \in X.$$

Definition 2.10. A map $M : X \to 2^X$ is said to be maximal monotone if

(i) M is monotone
(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Proposition 2.4. *Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-maximal monotone mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ defined by*

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u)$$

is single-valued.

Theorem 2.1 ([1]). *Let X be a real Hilbert space, and let $M : X \to 2^X$ be a multivalued mapping. Then we have:*

(i) *$M : X \to 2^X$ is monotone if and only if the resolvent operator associated with M and defined by*

$$J^M_\rho(u) = (I + \rho M)^{-1}(u) \quad \forall\, u \in X,$$

is firmly nonexpansive.

(ii) *$M : X \to 2^X$ is maximal monotone if and only if the resolvent operator*

$$J^M_\rho(u) = (I + \rho M)^{-1}(u) \quad \forall\, u \in X$$

associated with M is firmly nonexpansive and $\mathrm{dom}(J^M_\rho) = X$, *where I is the identity mapping.*

Lemma 2.1. *Let X be a real Hilbert space, let $J : X \to X$ be a mapping on X, and let r be a positive constant. Then*

(i) *All (r)-firmly nonexpansive mappings are (r)-Lipschitz continuous.*
(ii) *If a map J is (r)-firmly nonexpansive, then $2J - I$ is nonexpansive for $0 < r < 1$, where I is the identity mapping.*
(iii) *If a map J is (r)-firmly nonexpansive, then $I - J$ is nonexpansive for $0 < r < 1$, where I is the identity mapping.*

Proof. (i) It follows from the definition of the (r)-firm nonexpansiveness.

(ii) Since J is (r)-firmly nonexpansive, for $x, y \in X$, we have

$$\begin{aligned}
\|(2J - I)(x) - (2J - I)(y)\|^2 &= \langle 2J(x) - 2J(y) - (x - y), 2J(x) - 2J(y) - (x - y)\rangle \\
&= \|2J(x) - 2J(y)\|^2 + \|x - y\|^2 - 2\langle 2J(x) - 2J(y), x - y\rangle \\
&\leq r\langle 2J(x) - 2J(y), x - y\rangle + \|x - y\|^2 - 2\langle 2J(x) - 2J(y), x - y\rangle \\
&\leq \langle 2J(x) - 2J(y), x - y\rangle + \|x - y\|^2 - 2\langle 2J(x) - 2J(y), x - y\rangle \\
&\leq \|(2J - I)(x) - (2J - I)(y)\|\,\|x - y\| \quad \text{for } 0 < r < 1.
\end{aligned}$$

(iii) Since J is (r)-firmly nonexpansive for $x, y \in X$, we have

$$\begin{aligned}
\|(I - J)(x) - (I - J)(y)\|^2 &= \langle x - y - (J(x) - J(y)), x - y - (J(x) - J(y))\rangle \\
&= \|J(x) - J(y)\|^2 + \|x - y\|^2 - 2\langle J(x) - J(y), x - y\rangle \\
&\leq r\langle J(x) - J(y), x - y\rangle + \|x - y\|^2 - 2\langle J(x) - J(y), x - y\rangle \\
&\leq \langle J(x) - J(y), x - y\rangle + \|x - y\|^2 - 2\langle J(x) - J(y), x - y\rangle \quad \text{for } 0 < r < 1 \\
&= \|x - y\|^2 - \langle J(x) - J(y), x - y\rangle \\
&= \langle (I - J)(x) - (I - J)(y), x - y\rangle. \quad \square
\end{aligned}$$

Lemma 2.2 ([1]). *Let X be a real Hilbert space, and let $J : X \to X$ be a mapping on X. Then*

(i) *All firmly nonexpansive mappings are nonexpansive.*
(ii) *A map J is firmly nonexpansive if and only if $2J - I$ is nonexpansive.*
(iii) *A map J is firmly nonexpansive if and only if $I - J$ is firmly nonexpansive, where I is the identity mapping.*

Note that a mapping $J : X \to X$ is $(1/r)$-firmly nonexpansive is equivalent to stating that J is (r)-cocoercive, that is

$$\langle u - v, J(u) - J(v)\rangle \geq r\|J(u) - J(v)\|^2 \quad \forall u, v \in X.$$

This clearly reflects the connection between the notions of the firm nonexpansiveness and cocoercivity.

3. Generalized Eckstein–Bertsekas proximal algorithm

In this section we deal with the generalized Eckstein–Bertsekas proximal point algorithm and its application to approximation solvability of the inclusion problem (1) based on the A-maximal monotonicity. Furthermore, some auxiliary results also connecting the A-maximal monotonicity and corresponding generalized resolvent operator are established, that generalize the results on the firm nonexpansiveness [1] and A-monotonicity [7], while the auxiliary results on H-maximal monotonicity and general maximal monotonicity are derived.

Lemma 3.1 (*The Representation Lemma*). *Let X be a real Hilbert space, and let $M : X \to 2^X$ be a set-valued mapping. If M is A-maximal monotone, then every element $z \in X$ can be represented exactly in one way as*

$$A(z) = A(x) + \rho y,$$

where ρ is a positive constant.

Lemma 3.2 ([1, *The Representation Lemma*]). *Let X be a real Hilbert space, and let $M : X \to 2^X$ be a set-valued mapping. If M is maximal monotone, then every element $z \in X$ can be represented exactly in one way as $x + \rho y$, where ρ is a positive constant.*

Lemma 3.3 ([1, *Theorem 2*]). *Let X be a real Hilbert space. Then $M : X \to 2^X$ is monotone if and only if the resolvent operator defined by*

$$J_\rho^M(u) = (I + \rho M)^{-1}(u) \quad \forall u \in X,$$

is firmly nonexpansive, where ρ is a positive constant. Furthermore, M is maximal monotone if and only if J_ρ^M is firmly nonexpansive and $\mathrm{dom}(J_\rho^M) = X$.

Lemma 3.4 ([7]). *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal monotone. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u) \quad \forall u \in X,$$

is $\frac{1}{r-\rho m}$- Lipschitz continuous.

Lemma 3.5 ([11]). *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be H-maximal monotone. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X,$$

is $(1/r)$- Lipschitz continuous.

Lemma 3.6. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal monotone. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u) \quad \forall u \in X,$$

satisfies

$$\|J_{\rho,A}^M(u) - J_{\rho,A}^M(v)\|^2 \leq \frac{1}{r - \rho m}\langle u - v, J_{\rho,A}^M(u) - J_{\rho,A}^M(v)\rangle.$$

Consequently, we have

$$\|J_{\rho,A}^M(u) - J_{\rho,A}^M(v)\| \leq \frac{1}{r - \rho m}\|u - v\|,$$

and

$$\langle u - v, J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\rangle \leq \frac{1}{r-\rho m}\|u - v\|^2.$$

Proof. For any $u, v \in X$, it follows from the definition of the resolvent operator $J^M_{\rho,A}$ that

$$\frac{1}{\rho}[u - A(J^M_{\rho,A}(u))] \in M(J^M_{\rho,A})(u),$$

and

$$\frac{1}{\rho}[v - A(J^M_{\rho,A}(v))] \in M(J^M_{\rho,A})(v).$$

Since M is A-maximal monotone, we have

$$\frac{1}{\rho}\langle u - v - [A(J^M_{\rho,A}(u)) - A(J^M_{\rho,A}(v))], J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\rangle \geq -m\|J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\|^2 \tag{5}$$

In light of (5), we have

$$\begin{aligned}\langle u - v, J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\rangle &\geq \langle A(J^M_{\rho,A}(u)) - A(J^M_{\rho,A}(v)), J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\rangle - \rho m\|J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\|^2\\ &\geq (r - \rho m)\|J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\|^2.\end{aligned}$$

Therefore,

$$\|J^M_{\rho,H}(u) - J^M_{\rho,H}(v)\|^2 \leq \frac{1}{r-\rho m}\langle u - v, J^M_{\rho,A}(u), J^M_{\rho,A}(v)\rangle. \quad \square$$

Lemma 3.7. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal monotone. Then $I - J^M_{\rho,A}$ is $([1 + \frac{1}{r-\rho m}]^2[1 - \frac{1}{r-\rho m}]^{-1})$- firmly nonexpansive for $1 < r - \rho m$.*

Proof. Since $J^M_{\rho,A}$ is $\frac{1}{r-\rho m}$-Lipschitz continuous for $x, y \in X$, we have

$$\|J^M_{\rho,A}(x) - J^M_{\rho,A}(y)\| \leq \frac{1}{r-\rho m}\|x - y\|.$$

Furthermore, we have

$$\|(I - J^M_{\rho,A})(x) - (I - J^M_{\rho,A})(y)\| \leq \left[1 + \frac{1}{r-\rho m}\right]\|x - y\|.$$

Therefore, it follows that

$$\begin{aligned}\langle (I - J^M_{\rho,A})(x) - (I - J^M_{\rho,A})(y), x - y\rangle &\geq \left[1 - \frac{1}{r-\rho m}\right]\|x - y\|^2\\ &\geq \left[1 + \frac{1}{r-\rho m}\right]^{-2}\left[1 - \frac{1}{r-\rho m}\right]\|(I - J^M_{\rho,A})(x) - (I - J^M_{\rho,A})(y)\|^2.\end{aligned}$$

Hence,

$$\|(I - J^M_{\rho,A})(x) - (I - J^M_{\rho,A})(y)\|^2 \leq \left[1 + \frac{1}{r-\rho m}\right]^2\left[1 - \frac{1}{r-\rho m}\right]^{-1}\langle (I - J^M_{\rho,A})(x) - (I - J^M_{\rho,A})(y), x - y\rangle. \quad \square$$

Theorem 3.1. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal monotone. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to (1).*
(ii) *For an $u \in X$, we have*

$$u = J^M_{\rho,A}(A(u)),$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Proof. It follows from the definition of the generalized resolvent operator corresponding to M. Note that Theorem 3.1 generalizes [1, Lemma 2] to the case of A-maximal monotone mappings. $\square$

Theorem 3.2. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be H-maximal monotone. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to (1).*
(ii) *For an $u \in X$, we have*

$$u = J_{\rho,H}^{M}(H(u)),$$

where

$$J_{\rho,H}^{M}(u) = (H + \rho M)^{-1}(u).$$

Theorem 3.3. *Let X be a real Hilbert space, let $I : X \to X$ denote the identity mapping on X, and let $M : X \to 2^X$ be maximal monotone. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to (1).*
(ii) *For an $u \in X$, we have*

$$u = J_{\rho}^{M}(u),$$

where

$$J_{\rho}^{M}(u) = (I + \rho M)^{-1}(u).$$

Lemma 3.8. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be A-maximal monotone. Then $(I - J_{\rho}^{M,A} oA)$ is $([1 + \frac{s}{r-\rho m}]^2[1 - \frac{s}{r-\rho m}]^{-1})$- firmly nonexpansive for $s < r - \rho m$.*

Proof. The proof follows from the $(\frac{1}{r-\rho m})$-Lipschitz continuity [7] of $J_{\rho}^{M,A}$ and Lipschitz continuity of A. For $u, v \in X$, we have

$$\|J_{\rho,A}^{M}(A(u)) - J_{\rho,A}^{M}(A(v))\| \leq \frac{s}{r - \rho m}\|u - v\|,$$

and

$$\|(I - J_{\rho,A}^{M} oA)(u) - (I - J_{\rho}^{M} oA)(v)\| \leq \left[1 + \frac{s}{r - \rho m}\right]\|u - v\|.$$

This implies that

$$\begin{aligned}\langle (I - J_{\rho}^{M} oA)(u) - (I - J_{\rho}^{M} oA)(v), u - v\rangle &\geq \left[1 - \frac{s}{r - \rho m}\right]\|u - v\|^2 \\ &\geq \left[1 + \frac{s}{r - \rho m}\right]^{-2}\left[1 - \frac{s}{r - \rho m}\right]\|(I - J_{\rho}^{M} oA)(u) - (I - J_{\rho}^{M} oA)(v)\|^2.\end{aligned}$$

Hence, we have

$$\|(I - J_{\rho}^{M} oA)(u) - (I - J_{\rho}^{M} oA)(v)\|^2 \leq \left[1 + \frac{s}{r - \rho m}\right]^{2}\left[1 - \frac{s}{r - \rho m}\right]^{-1}\langle (I - J_{\rho}^{M} oA)(u) - (I - J_{\rho}^{M} oA)(v), u - v\rangle,$$

where $s < r - \rho m$. $\square$

Lemma 3.9. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be H-maximal monotone. Then $(I - J_{\rho}^{M,H} oH)$ is $([1 + \frac{s}{r}]^2[1 - \frac{s}{r}]^{-1})$- firmly nonexpansive for $s < r$.*

Next, we introduce the Generalized Eckstein–Bertsekas Proximal point algorithm.

Algorithm 3.1. Let $A : X \to X$ be a single-valued mapping, let $M : X \to 2^X$ be a set-valued A-maximal monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{6}$$

and y^k satisfies

$$\|y^k - J_{\rho_k,A}^{M}(A(x^k))\| \leq \epsilon_k,$$

where $J^M_{\rho_k,A} = (A + \rho_k M)^{-1}$, and

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

are scalar sequences.

Algorithm 3.2. Let $H : X \to X$ be a single-valued mapping, let $M : X \to 2^X$ be a set-valued H-maximal monotone mapping on X with $0 \in \mathrm{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{7}$$

and y^k satisfies

$$\|y^k - J^M_{\rho_k,H}(H(x^k))\| \leq \epsilon_k,$$

where $J^M_{\rho_k,H} = (H + \rho_k M)^{-1}$, and

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

are scalar sequences.

Algorithm 3.3. Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in \mathrm{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{8}$$

and y^k satisfies

$$\|y^k - J^M_{\rho_k}(x^k)\| \leq \epsilon_k,$$

where $J^M_{\rho_k} = (I + \rho_k M)^{-1}$, and

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

are scalar sequences.

Theorem 3.4. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be A-maximal monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by Algorithm 3.1*

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{9}$$

such that y^k satisfies

$$\|y^k - J^M_{\rho_k,A}(A(x^k))\| \leq \epsilon_k,$$

where $J^M_{\rho_k,A} = (A + \rho_k M)^{-1}$, and the sequences

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

satisfy

$$e_1 = \sum_{k=0}^{\infty} \epsilon_k < \infty, \qquad \Delta_1^* = \inf \alpha_k > 0, \qquad \Delta_2^* = \sup \alpha_k < 2, \quad \text{and} \quad \rho = \inf \rho_k > 0.$$

Then the sequence $\{x^k\}$ converges weakly to a solution of (1) for $s < r - \rho m$.

Proof. Suppose that x^* is a zero of M. For all $k \geq 0$, we set

$$J_k^* = I - J^M_{\rho_k,A} o A.$$

Then, in light of Lemma 3.8, J_k^* is $([1 + \frac{s}{r-\rho m}]^2[1 - \frac{s}{-\rho m}]^{-1})$- firmly nonexpansive for $s < r - \rho m$, and as a consequence

$$\langle x^k - x^*, J_k^*(x^k)\rangle \geq \left[1 + \frac{s}{r - \rho_k m}\right]^{-2} \left[1 - \frac{s}{r - \rho_k m}\right] \|J_k^*(x^k)\|^2.$$

Furthermore, from Theorem 3.1, it follows that any solution to (1) is a fixed point of $J^M_{\rho_k,A}oA$, and hence a zero of J^*_k. For all $k \geq 0$, we express

$$\begin{aligned} z^{k+1} &= (1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)) \\ &= (I - \alpha_k J^*_k)(x^k). \end{aligned}$$

Next, we estimate

$$\begin{aligned} \|z^{k+1} - x^*\|^2 &= \|(1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)) - x^*\|^2 \\ &= \|x^k - x^* - \alpha_k J^*_k(x^k)\|^2 \\ &= \|x^k - x^*\|^2 - 2\alpha_k\langle x^k - x^*, J^*_k(x^k)\rangle + \alpha_k^2\|J^*_k(x^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\left[1 + \frac{s}{r - \rho_k m}\right]^{-2}\left[1 - \frac{s}{r-\rho_k m}\right]\alpha_k\|J^*_k(x^k)\|^2 + \alpha_k^2\|J^*_k(x^k)\|^2 \\ &= \|x^k - x^*\|^2 - \left(2\left[1 + \frac{s}{r - \rho_k m}\right]^{-2}\left[1 - \frac{s}{r-\rho_k m}\right] - \alpha_k\right)\alpha_k\|J^*_k(x^k)\|^2 \\ &= \|x^k - x^*\|^2 - [2\theta - \alpha_k]\alpha_k\|J^*_k(x^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - \triangle^*_1(2\theta - \triangle^*_2)\|J^*_k(x^k)\|^2, \end{aligned}$$

where $\theta = [1 + \frac{s}{r-\rho m}]^{-2}[1 - \frac{s}{r-\rho m}]$ for $s < r - \rho_k m$.

Since $\triangle^*_1(2\theta - \triangle^*_2) > 0$, it implies that

$$\|z^{k+1} - x^*\| \leq \|x^k - x^*\|.$$

It further follows that

$$\begin{aligned} \|x^{k+1} - z^{k+1}\| &= \|(1-\alpha_k)x^k + \alpha_k y^k - [(1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k))]\| \\ &= \|\alpha_k(y^k - J^M_{\rho_k,A}(A(x^k)))\| \\ &\leq \alpha_k\epsilon_k. \end{aligned}$$

Now we find the estimate that leads to the boundedness of the sequence $\{x^k\}$.

$$\|x^{k+1} - x^*\| \leq \|z^{k+1} - x^*\| + \|x^{k+1} - z^{k+1}\| \leq \|x^k - x^*\| + \alpha_k\epsilon_k. \tag{10}$$

Combining inequality (10) for all k, we get

$$\|x^{k+1} - x^*\| \leq \|x^0 - x^*\| + \sum_{j=0}^{k}\alpha_j\epsilon_j \leq \|x^0 - x^*\| + 2e_1. \tag{11}$$

Therefore, the sequence $\{x^k\}$ is bounded.

To establish the weak convergence of the sequence $\{x^k\}$, we examine the estimate

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|z^{k+1} - x^* + x^{k+1} - z^{k+1}\|^2 \\ &\leq \|x^k - x^*\|^2 - \triangle^*_1(2\theta - \triangle^*_2)\|J^*_k(x^k)\|^2 + 2\alpha_k\epsilon_k(\|x^0 - x^*\| + 2e_1) + \alpha_k^2\epsilon_k^2. \end{aligned} \tag{12}$$

Using the summability of the sequence $\{\epsilon_k\}$, we have

$$e_2 = \sum_{k=0}^{\infty}\epsilon_k^2 < \infty.$$

As a result, we find

$$\|x^{k+1} - x^*\|^2 \leq \|x^0 - x^*\|^2 + 4e_1(\|x^0 - x^*\| + 2e_1) + 4e_2 - \triangle^*_1(2\theta - \triangle^*_2)\sum_{j=0}^{k}\|J^*_j(x^j)\|^2. \tag{13}$$

As $k \to \infty$, we have that

$$\sum_{j=0}^{k}\|J^*_j(x^j)\|^2 < \infty \Rightarrow \lim_{k\to\infty} J^*_k(x^k) = 0.$$

Applying Lemma 3.1, there is a unique element $(u^k, v^k) \in M$ represented by $A(u^k) + \rho_k v^k = A(x^k)$ for all k. Since $u^k = (J^M_{\rho_k,A} oA)(x^k)$ and $\lim_{k\to\infty} J^*_k(x^k) = 0$, it implies that $x^k - u^k \to 0$. Therefore, in light of the (s)-Lipschitz continuity of A, it follows that $A(x^k) - A(u^k) \to 0$. On the other hand, we have, since the sequence $\{\rho_k\}$ is bounded away from zero, that

$$\lim_{k\to\infty} \frac{J^*_k(x^k)}{\rho_k} = \lim_{k\to\infty} v^k = 0.$$

In light of above arguments, the sequence $\{x^k\}$ is bounded, and hence it must have at least one weak cluster point, say x'. Let $\{x^{k(j)}\}$ be a subsequence of $\{x^k\}$ such that $x^{k(j)}$ converges weakly to x'. Since $x^k - u^k \to 0$, it implies $u^{k(j)}$ also converges weakly to x'. Let some $(u, v) \in M$. Then the A-maximal monotonicity of M implies that

$$\langle u - u^k, v - v^k\rangle \geq -m\|u - u^k\|^2 \quad \text{for all } k \geq 0.$$

It follows that

$$\langle u - x', v - 0\rangle \geq -m\|u - x'\|^2 \quad \text{for all } k \geq 0.$$

Since M is A-maximal monotone, and (u, v) is arbitrary, it follows that $(x', 0) \in M$, that is, x' is a solution to (1).

To complete the proof, we still need to show the uniqueness of the weak cluster point of the sequence $\{x^k\}$. Assume that x_1 and x_2 are two distinct weak cluster points of the sequence $\{x^k\}$ (and hence both are zeros of M). It follows from Theorem 3.1 that

$$\begin{aligned} x_1 &= J^M_{\rho_k,A}(A(x_1)),\\ x_2 &= J^M_{\rho_k,A}(A(x_2)).\\ \|x_1 - x_2\| &= \|J^M_{\rho_k,A}(A(x_1)) - J^M_{\rho_k,A}(A(x_2))\|\\ &\leq \frac{s}{r - \rho m}\|x_1 - x_2\|. \end{aligned}$$

Hence, we have

$$\left[1 - \frac{s}{r - \rho m}\right]\|x_1 - x_2\| \leq 0,$$

and this shows that

$$x_1 = x_2 \quad \text{for } s < r - \rho m \quad \square$$

Corollary 3.1. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be H-maximal monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by Algorithm 3.2*

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{14}$$

such that y^k satisfies

$$\|y^k - J^M_{\rho_k,H}(H(x^k))\| \leq \epsilon_k,$$

where $J^M_{\rho_k,H} = (H + \rho_k M)^{-1}$, and the sequences

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

satisfy

$$e_1 = \sum_{k=0}^{\infty} \epsilon_k < \infty, \qquad \triangle^*_1 = \inf \alpha_k > 0, \qquad \triangle^*_2 = \sup \alpha_k < 2, \quad \text{and} \quad \rho = \inf \rho_k > 0.$$

Then the sequence $\{x^k\}$ converges weakly to a solution of (1) for $s < r$.

Proof. The proof is similar to that Theorem 3.4, but we include a brief sketch of the proof for the sake of the completeness. Let x^* be a zero of M. For all $k \geq 0$, we set

$$J^*_k = I - J^M_{\rho_k,H} oH.$$

Then, in light of Lemma 3.9, J^*_k is $([1 + \frac{s}{r}]^2[1 - \frac{s}{r}]^{-1})$-firmly nonexpansive for $s < r$. Furthermore, from Theorem 3.2, it follows that any solution to (1) is a fixed point of $J^M_{\rho_k,A} oH$, and hence a zero of J^*_k. For all $k \geq 0$, we express

$$\begin{aligned} z^{k+1} &= (1 - \alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(H(x^k))\\ &= (I - \alpha_k J^*_k)(x^k). \end{aligned}$$

Next, we estimate

$$\begin{aligned}\|z^{k+1}-x^*\|^2 &= \|(1-\alpha_k)x^k+\alpha_k J^M_{\rho_k,H}(H(x^k))-x^*\|^2\\ &= \|x^k-x^*-\alpha_k J^*_k(x^k)\|^2\\ &= \|x^k-x^*\|^2-2\alpha_k\langle x^k-x^*, J^*_k(x^k)\rangle+\alpha_k^2\|J^*_k(x^k)\|^2\\ &\le \|x^k-x^*\|^2-2\left[1+\frac{s}{r}\right]^{-2}\left[1-\frac{s}{r}\right]\alpha_k\|J^*_k(x^k)\|^2+\alpha_k^2\|J^*_k(x^k)\|^2\\ &= \|x^k-x^*\|^2-\left(2\left[1+\frac{s}{r}\right]^{-2}\left[1-\frac{s}{r}\right]-\alpha_k\right)\alpha_k\|J^*_k(x^k)\|^2\\ &= \|x^k-x^*\|^2-([2\theta^*-\alpha_k]\alpha_k)\|J^*_k(x^k)\|^2\\ &\le \|x^k-x^*\|^2-\triangle_1^*(2\theta^*-\triangle_2^*)\|J^*_k(x^k)\|^2,\end{aligned}$$

where $\theta^*=([1+\frac{s}{r}]^{-2}[1-\frac{s}{r}])$ for $s<r$.

Since $\triangle_1^*(2\theta^*-\triangle_2^*)>0$, it implies that

$$\|z^{k+1}-x^*\|\le\|x^k-x^*\|.$$

It further follows from the above arguments that

$$\|x^{k+1}-z^{k+1}\| \tag{15}$$

$$=\|(1-\alpha_k)x^k+\alpha_k y^k-[(1-\alpha_k)x^k+\alpha_k J^M_{\rho_k,A}(H(x^k))]\| \tag{16}$$

$$=\|\alpha_k(y^k-J^M_{\rho,A}(H(x^k)))\| \tag{17}$$

$$\le\alpha_k\epsilon_k. \tag{18}$$

Now we examine the boundedness of the sequence $\{x^k\}$.

$$\begin{aligned}\|x^{k+1}-x^*\| &\le \|z^{k+1}-x^*\|+\|x^{k+1}-z^{k+1}\|\\ &\le \|x^k-x^*\|+\alpha_k\epsilon_k.\end{aligned}\tag{19}$$

Next, combining inequality (19) for all k, we get

$$\|x^{k+1}-x^*\|\le\|x^0-x^*\|+\sum_{j=0}^{k}\alpha_j\epsilon_j\le\|x^0-x^*\|+2e_1. \tag{20}$$

Therefore, the sequence $\{x^k\}$ is bounded. To establish the weak convergence of the sequence $\{x^k\}$, estimate

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &= \|z^{k+1}-x^*+x^{k+1}-z^{k+1}\|^2\\ &\le \|x^k-x^*\|^2-\triangle_1^*(2\theta-\triangle_2^*)\|J^*_k(x^k)\|^2+2\alpha_k\epsilon_k(\|x^0-x^*\|+2e_1)+\alpha_k^2\epsilon_k^2.\end{aligned}\tag{21}$$

Based on the summability of the sequence $\{\epsilon_k\}$, we have

$$e_2=\sum_{k=0}^{\infty}\epsilon_k^2<\infty.$$

As a result, we find

$$\|x^{k+1}-x^*\|^2\le\|x^0-x^*\|^2+4e_1(\|x^0-x^*\|+2e_1)+4e_2-\triangle_1^*(2\theta-\triangle_2^*)\sum_{j=0}^{k}\|J^*_j(x^j)\|^2. \tag{22}$$

As $k\to\infty$, we have that

$$\sum_{j=0}^{k}\|J^*_j(x^j)\|^2<\infty\Rightarrow\lim_{k\to\infty}J^*_k(x^k)=0.$$

Applying Lemma 3.1 when M is an *H-maximal* monotone mapping, there is a unique element $(u^k, v^k)\in M$ represented by $H(u^k)+\rho_k v^k=H(x^k)$ for all k. Since $u^k=(J^M_{\rho_k}oH)(x^k)$ and $\lim_{k\to\infty}J^*_k(x^k)=0$, it implies that $x^k-u^k\to 0$. Therefore, in light of the (s)-Lipschitz continuity of H, it follows that $H(x^k)-H(u^k)\to 0$. On the other hand, it further follows, since the sequence $\{\rho_k\}$ is bounded away from zero, that

$$\lim_{k\to\infty}\frac{J^*_k(x^k)}{\rho_k}=\lim_{k\to\infty}v^k=0.$$

In light of above arguments, the sequence $\{x^k\}$ is bounded, and hence it must have at least one weak cluster point, say x'. Let $\{x^{k(j)}\}$ be a subsequence of $\{x^k\}$ such that $x^{k(j)}$ converges weakly to x'. Since $x^k - u^k \to 0$, it implies $u^{k(j)}$ also converges weakly to x'. Let some $(u, v) \in M$. Then the H-maximal monotonicity of M implies that

$$\langle u - u^k, v - v^k \rangle \geq 0 \quad \text{for all } k \geq 0.$$

It follows that

$$\langle u - x', v - 0 \rangle \geq 0 \quad \text{for all } k \geq 0.$$

Since M is H-maximal monotone, and (u, v) is arbitrary, it follows that $(x', 0) \in M$, that is, x' is a solution to (1). □

4. General remark

(a) The generalized Eckstein–Bertsekas proximal point algorithm based on A-maximal monotone mappings can be applied to the relaxed Douglas–Rachford splitting methods [5,14,1] since it is easier to apply the generalized Eckstein–Bertsekas proximal point algorithm and other related methods to map M by splitting $M = S + T$, where S and T are A-maximal monotone mappings. Let $A : X \to X$ be an operators on X. There has always been a serious difficulty in evaluating the inverses of the operators of form $I + \rho M$ and $A + \rho M$ for $\rho > 0$, especially for maximal monotone and A-maximal monotone operators. This paved the way to the splitting algorithm because the splitting $M = S + T$ brings easier evaluations of resolvent operators $J^S_{A,\rho} = (A + \rho S)^{-1}$ and $J^T_{A,\rho} = (A + \rho T)^{-1}$ than $J^M_{A,\rho} = (A + \rho M)^{-1}$. Here we generalize the Douglas–Rachford splitting iteration to the case of the A-maximal monotonicity framework as follows: consider two A-maximal monotone operators $S, T : X \to 2^X$ and fix $\rho > 0$. The sequence $\{z^k\}_{k=0}^{\infty}$ is said to satisfy the generalized Douglas–Rachford iteration for ρ, S and T if

$$z^{k+1} = J^S_{A,\rho}((2J^T_{A,\rho} - I)(z^k)) + (I - J^T_{A,\rho})(z^k).$$

Based on Lemma 3.1, let any sequence satisfies the generalized Douglas–Rachford iteration, and let (x^k, t^k) be the unique element of T such that

$$A(x^k) + \rho t^k = A(z^k).$$

Then for all k, we have (using the definition of $J^T_{A,\rho}$) that

$$(I - J^T_{A,\rho})(z^k) = A(x^k) + \rho t^k - A(x^k) = \rho t^k,$$

and

$$(2J^T_{A,\rho} - I)(z^k) = A(x^k) - \rho t^k.$$

In a similar manner, we have (using the definition of $J^S_{A,\rho}$) that if $(y^k, s^k) \in S$, then $J^S_{A,\rho}(A(y^k) + \rho s^k) = A(y^k)$.

Furthermore, based on the work of Pennanen [6], the generalized Eckstein–Bertsekas proximal point algorithm can also be applied to the case of local convergence and local monotonicity (that is, without global monotonicity restrictions) and more splitting methods.

(b) It seems that if one considers the following inclusion problem: find a solution to

$$0 \in M(u) + N(u), \tag{23}$$

where $M : X \to 2^X$ is a set-valued mapping on X, and $N : X \to X$ is another mapping on X, then the solvability of this problem is similar to that of Theorem 3.4, but here we expect a linear convergence of the sequence based on Algorithm 4.1 and Theorem 4.1.

Algorithm 4.1. Let $A : X \to X$ be a single-valued mapping, let $M : X \to 2^X$ be a set-valued A-maximal monotone mapping on X with $0 \in \text{range}(M)$, and let $N : X \to X$ be another single-valued mapping on X. Let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \tag{24}$$

and y^k satisfies

$$\|y^k - (J^M_{\rho_k,A} o(A - \rho_k N))(x^k)\| \leq \delta_k \|y^k - x^k\|,$$

and $\delta_k \to 0$ with $J^M_{\rho_k,A} = (A + \rho_k M)^{-1}$, and

$$\{\epsilon_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

are scalar sequences.

Theorem 4.1. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal monotone. Let $N : X \to X$ be a cocoercively monotone mapping on X. Then the following statements are equivalent:*

(i) *An element $u \in X$ is a solution to (23).*
(ii) *For an $u \in X$, we have*

$$u = (J^M_{\rho,A} o(A - \rho N))(u),$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

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